ISRAEL JOURNAL OF MATHEMATICS **161** (2007), 1–15
DOI: 10.1007/s11856-007-0069-8

INJECTIVE LINEAR CELLULAR AUTOMATA AND SOFIC GROUPS

BY

TULLIO CECCHERINI-SILBERSTEIN

Dipartimento di Ingegneria, Università del Sannio
C.so Garibaldi 107, 82100 Benevento, Italy
e-mail: tceccher@mat.uniroma1.it

AND

MICHEL COORNAERT

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS
7 rue René Descartes, 67084 Strasbourg Cedex, France
e-mail: coornaert@math.u-strasbg.fr

ABSTRACT

Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and let G be a sofic group. We show that every injective linear cellular automaton $\tau\colon V^G \to V^G$ is surjective. As an application, we obtain a new proof of the stable finiteness of group rings of sofic groups, a result previously established by G. Elek and A. Szabó using different methods.

1. Introduction

Sofic groups were introduced by M. Gromov in [Gro] under the name of **initially subamenable** groups. The sofic terminology is due to B. Weiss [Wei] (see Section 2 below for the definition of soficity). The class of sofic groups contains, in particular, all residually amenable groups and it is closed under direct products, free products, taking subgroups and extensions by amenable groups [ES2].

Weiss proved in [Wei] that every sofic group is surjunctive in the sense of Gottschalk [Got]. We recall that a group G is said to be **surjunctive** if, for any finite alphabet A, every injective cellular automaton $\tau\colon A^G \to A^G$ over G

Received October 17, 2005 and in revised form January 4, 2007

is surjective (see Section 2 for basic facts about cellular automata). It should be pointed out that there is no known example of a non-surjunctive group nor even of a non-sofic group up to now.

A **linear cellular automaton** is a cellular automaton $\tau\colon V^G \to V^G$, where the alphabet is a vector space V over some field $\mathbb{K}$ and τ is $\mathbb{K}$-linear.

In analogy with the classical finite alphabet case, we introduce the following

Definition 1.1: A group G is said to be **L-surjunctive** if, for any field $\mathbb{K}$ and any finite-dimensional vector space V over $\mathbb{K}$, every injective linear cellular automaton $\tau\colon V^G \to V^G$ is surjective.

In [CeC, Theorem 1.2 and 1.3] we showed, among other things, that residually finite groups and amenable groups are L-surjunctive. Observe that residually finite groups and amenable groups are residually amenable and hence sofic. Thus the following theorem, which is the main result of this paper, is a generalization.

THEOREM 1.2: *Every sofic group is L-surjunctive.*

Let us note that both surjunctivity and L-surjunctivity of sofic groups are particular cases of a very general surjunctivity theorem in [Gro] for projective systems over initially amenable graphs.

The fact that a group G is L-surjunctive or not can be seen on the algebraic structure of its group rings $\mathbb{K}[G]$. This is related to a famous conjecture of Kaplansky in group ring theory. Recall that a ring R with unit element 1_R is said to be **directly finite** if one-sided inverses in R are also two-sided inverses, i.e., $ab = 1_R$ implies $ba = 1_R$ for $a, b \in R$. The ring R is said to be **stably finite** if the ring $\mathrm{Mat}_d(R)$ of $d \times d$ matrices with coefficients in R is directly finite for all integers $d \geq 1$. Kaplansky [K] conjectured that, for any group G and any field $\mathbb{K}$, the group ring $\mathbb{K}[G]$ is directly finite. We will prove the following result.

THEOREM 1.3: *A group G is L-surjunctive if and only if the group ring $\mathbb{K}[G]$ is stably finite for all field $\mathbb{K}$.*

As an immediate consequence of Theorem 1.2 and 1.3, we deduce the following result, which has been previously established by G. Elek and A. Szabó [ES1] using different methods (their proof involves embeddings of the group rings into continuous von Neumann regular rings).

COROLLARY 1.4: *Let G be a sofic group and $\mathbb{K}$ any field. Then the group ring $\mathbb{K}[G]$ is stably finite. In particular, $\mathbb{K}[G]$ is directly finite.*

The paper is structured as follows. In Section 2 we give some preliminaries and notation. In particular, we introduce the notion of a linear subshift and that of a linear cellular automaton between linear subshifts. We then prove in Section 3 that the inverse of a bijective linear cellular automaton between linear subshifts over a countable group is also a (linear) cellular automaton. In Section 4 we show that linear cellular automata of dimension d over a group G with coefficients in a field $\mathbb{K}$ can be represented by $d \times d$ matrices over the group ring $\mathbb{K}[G]$. We use this matrix representation to show that L-surjunctivity is equivalent to the stable finiteness of the group rings (Theorem 1.3). In the last section we prove that soficity implies L-surjunctivity (Theorem 1.2).

2. Background, definitions and notation

2.1. Labeled graphs. Let S be a set. An S-**labeled graph** is a triple (Q, E, λ), where Q is a set, called the set of **vertices**, E is a symmetric subset of $Q \times Q$, called the set of **edges**, and λ is a map from E to S, called the **labeling map**. An S-labeled graph is said to be **finite** if its set of vertices is finite. We shall often identify a labeled graph with its underlying set of vertices if the labeled graph structure is understood.

Let (Q, E, λ) be an S-labeled graph. We shall view every subset $Q' \subset Q$ as an S-labeled graph (Q', E', λ') with set of edges $E' = E \cap (Q' \times Q')$ and labeling map $\lambda' = \lambda|_{E'}$. Given $q \in Q$ and $r \in \mathbb{N}$, we define the ball $B(q, r) \subset Q$ by $B(q, r) = \{q' \in Q : d(q, q') \leq r\}$, where d is the graph distance ($d(q, q') =$ minimal length of an edge path joining q and q' if there are such paths or ∞ otherwise).

Let (Q, E, λ) and (Q', E', λ') be S-labeled graphs. One says that (Q, E, λ) and (Q', E', λ') are **isomorphic** if there is a bijection $\varphi\colon Q \to Q'$ such that $(q_1, q_2) \in E$ if and only if $(\varphi(q_1), \varphi(q_2)) \in E'$ and $\lambda(q_1, q_2) = \lambda'(\varphi(q_1), \varphi(q_2))$ for all $(q_1, q_2) \in E$. Two vertices $q \in Q$ and $q' \in Q'$ are said to be r-**equivalent**, and we write $q \sim_r q'$, if there is an isomorphism of S-labeled graphs $\phi\colon B(q, r) \to B(q', r)$ such that $\phi(q) = q'$.

Consider now a group G with a symmetric generating subset $S \subset G$. The **Cayley graph** associated with (G, S) is the S-labeled graph (Q, E, λ), where $Q = G$, $E = \{(g, gs) : g \in G, s \in S\}$ and $\lambda\colon E \to S$ is defined by $\lambda(g, gs) = s$. We note that, for $g \in G$, left multiplication by g induces an automorphism of the Cayley graph sending the identity element 1_G of G to g. This shows, in particular, that 1_G and g are r-equivalent for all $r \in \mathbb{N}$.

2.2. Sofic groups. Before treating the general case, we first give the definition of soficity for finitely generated groups.

Definition 2.1: Let G be a finitely generated group and let S be a finite symmetric generating subset of G. The group G is said to be **sofic** if for all $\varepsilon > 0$ and $r \in \mathbb{N}$ there exists a finite S-labeled graph (Q, E, λ) such that the set $Q(r) \subset Q$ defined by $Q(r) = \{q \in Q : q \sim_r 1_G\}$ (here 1_G is considered as a vertex in the Cayley graph associated with (G, S)) satisfies

$$|Q(r)| \geq (1 - \varepsilon)|Q| \tag{2.1}$$

where $| \cdot |$ denotes cardinality.

It can be shown (see [Wei]) that this definition does not depend on the choice of S and that it can be extended as follows.

Definition 2.2: A (not necessarily finitely generated) group G is said to be **sofic** if all of its finitely generated subgroups are sofic according to Definition 2.1.

2.3. Cellular automata. Let G be a group. Given a set A, called the **alphabet**, we consider the set A^G consisting of all maps $x \colon G \to A$, equipped with the right action of G defined by $A^G \times G \ni (x, g) \mapsto x^g \in A^G$, where $x^g(g') = x(gg')$ for all $g' \in G$.

A **cellular automaton** over G on the alphabet A is a map $\tau \colon A^G \to A^G$ satisfying the following condition: there exists a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$\tau(x)(g) = \mu(x^g|_M) \quad \text{for all } x \in A^G, g \in G, \tag{2.2}$$

where $x^g|_M$ denotes the restriction of x^g to M. Such a set M is called a **memory set** and μ is called a **local defining map** for τ.

Note that if M is a memory set for τ, then any finite set $M' \subset G$ containing M is also a memory set for τ, with local defining map $\mu' \colon A^{M'} \to A$ given by $\mu' = \mu \circ \pi_{M',M}$ where $\pi_{M',M} \colon A^{M'} \to A^M$ is the restriction map. Finally, observe that every cellular automaton $\tau \colon A^G \to A^G$ is G-**equivariant**, i.e., it satisfies $\tau(x^g) = \tau(x)^g$ for all $g \in G$ and $x \in A^G$.

2.4. Linear cellular automata. Let G be a group and let V be a vector space over a field $\mathbb{K}$. A **linear cellular automaton** over G with alphabet space V is a cellular automaton $\tau \colon V^G \to V^G$ which is $\mathbb{K}$-linear. We will use the following characterization of linear cellular automata.

Lemma 2.3: *Let G be a group and let V be a vector space over a field $\mathbb{K}$. Let M be a finite subset of G. For a map $\tau \colon V^G \to V^G$, the following conditions are equivalent:*

(a) *τ is a linear cellular automaton with memory set M;*
(b) *there exists a family $(u_m)_{m\in M}$ of $\mathbb{K}$-linear maps $u_m\colon V \to V$ such that, for all $x \in V^G$,*

$$\tau(x)(g) = \sum_{m\in M} u_m(x(gm)) \quad \textit{for all } g \in G. \tag{2.3}$$

Proof: If τ satisfies (b), then τ is $\mathbb{K}$-linear and $\tau(x)(g) = \mu(x^g|_M)$, where $\mu\colon V^M \to V$ is given by $\mu(y) = \sum_{m\in M} u_m(y_m)$ for all $y = (y_m)_{m\in M} \in V^M$. Thus (b) implies (a).

Conversely, suppose that τ is a linear cellular automaton with memory set M and local defining map $\mu\colon V^M \to V$. Clearly, μ is $\mathbb{K}$-linear. Therefore, there exist $\mathbb{K}$-linear maps $u_m\colon V \to V$, $m \in M$, such that $\mu(y) = \sum_{m\in M} u_m(y_m)$ for all $y = (y_m)_{m\in M} \in V^M$. Thus we have

$$\tau(x)(g) = \mu(x^g|_M) = \sum_{m\in M} u_m(x(gm)).$$

this shows that (a) implies (b). ∎

Consider the group algebra $\mathbb{K}[G]$ (see Section 5) and let us equip the vector space V^G with the structure of right $\mathbb{K}[G]$-module induced by the right action of G on V^G. Then note that the set $\mathrm{LCA}(G,V)$ consisting of all linear cellular automata $\tau\colon V^G \to V^G$ is a subalgebra of the $\mathbb{K}$-algebra of endomorphisms of the right $\mathbb{K}[G]$-module V^G.

2.5. Surjunctivity. Let A be a set, G a group, and $\tau\colon A^G \to A^G$ a cellular automaton over G with memory set M and local defining map $\mu\colon A^M \to A$. Let H be a subgroup of G containing M and consider the map $\tau_H\colon A^H \to A^H$ defined by $\tau_H(x)(h) = \mu(x^h|_M)$ for all $x \in A^H$ and $h \in H$. Then τ_H is a cellular automaton over H, which is called the **restriction** of τ to H ([CeC]). We have the following relations between a cellular automaton and its restriction to a subgroup containing a memory set.

Proposition 2.4: *The cellular automaton τ is injective (resp. surjective) if and only if its restriction τ_H is injective (resp. surjective). Moreover, if A is a vector space, then τ is linear if and only if τ_H is linear.*

Proof: Let us fix a set of representatives $T \subset G$ for the left cosets of H in G so that $G = \coprod_{t\in T} tH$.

Suppose first that τ is injective. Let $x', y' \in A^H$ such that $\tau_H(x') = \tau_H(y')$. Extend x' and y' to x and y in A^G by setting $x(th) = x'(h)$ and $y(th) = y'(h)$

for all $h \in H$ and $t \in T$. As

$$\tau(x)(th) = \mu(x^{th}|_M) = \mu((x')^h|_M) = \tau_H(x')(h)$$

and similarly $\tau(y)(th) = \tau_H(y')(h)$ for all $h \in H$ and $t \in T$, we have $\tau(x) = \tau(y)$. It follows that $x = y$, by injectivity of τ. But then we get $x' \equiv x|_H = y|_H \equiv y'$. This shows that τ_H is also injective.

Conversely, suppose that τ_H is injective. Let $x, y \in A^G$ be such that $\tau(x) = \tau(y)$. For each $t \in T$, define $x_t, y_t \in A^H$ by setting $x_t(h) = x(th)$ and $y_t(h) = y(th)$ for all $h \in H$. We have

$$\tau_H(x_t)(h) = \mu((x_t)^h|_M) = \mu(x^{th}|_M) = \tau(x)(th)$$

and similarly $\tau_H(y_t)(h) = \tau(y)(th)$. It follows that $\tau_H(x_t) = \tau_H(y_t)$ for all $t \in T$. Thus $x_t = y_t$ for all t, by injectivity of τ_H. This clearly gives $x = y$ and injectivity of τ follows.

Suppose now that τ is surjective. Let $y' \in A^H$ and extend it arbitrarily to some element $y \in A^G$. By hypothesis, we can find $x \in A^G$ such that $\tau(x) = y$. But then $x' := x|_H$ satisfies

$$\tau_H(x')(h) = \mu((x')^h|_M) = \mu(x^h|_M) = \tau(x)(h) = y(h) = y'(h)$$

for all $h \in H$ i.e., $\tau_H(x') = y'$. Thus τ_H is surjective.

Conversely, suppose that τ_H is surjective. Let $y \in A^G$. For each $t \in T$, consider the element $y_t \in A^H$ defined by $y_t(h) = y(th)$ and choose $x_t \in A^H$ such that $\tau_H(x_t) = y_t$. Define $x \in A^G$ by $x(th) = x_t(h)$. Then, for all $h \in H$ and $t \in T$, we have

$$\tau(x)(th) = \mu(x^{th}|_M) = \mu((x_t)^h|_M) = \tau_H(x_t)(h) = y_t(h) = y(th),$$

that is $\tau(x) = y$. This shows that τ is surjective.

The proof of the last equivalence is left to the reader. ∎

From the preceding proposition, we easily deduce the following fact which, in the finite alphabet case, can be found in [Wei].

Corollary 2.5: *A group G is surjunctive (resp. L-surjunctive) if and only if all of its finitely generated subgroups are surjunctive (resp. L-surjunctive).*

2.6. Linear subshifts. Let G be a group and let V be a vector space over a field $\mathbb{K}$. We equip V^G with the Tychonov product topology where V is endowed with the discrete topology.

For a subset $X \subset V^G$ and a subset $\Omega \subset G$, we denote by $X|_\Omega$ the image of X under the natural projection (restriction map) $V^G \to V^\Omega$, that is, $X|_\Omega = \{x|_\Omega : x \in X\}$.

Definition 2.6: A subset $X \subset V^G$ is called a **linear subshift** if it is a closed G-invariant vector subspace of V^G.

We extend our definition of a linear cellular automaton to linear subshifts in the following way.

Definition 2.7: Let $X, Y \subset V^G$ be linear subshifts. A map $\tau\colon X \to Y$ is called a **linear cellular automaton** if it is the restriction $\tau = \overline{\tau}|_X$ of a linear cellular automaton $\overline{\tau}\colon V^G \to V^G$ such that $\overline{\tau}(X) \subset Y$.

We shall use the following Closure Lemma whose proof can be found in [CeC, Lemma 3.1.].

Lemma 2.8: *Let G be a countable group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau\colon V^G \to V^G$ be a linear cellular automaton. Then $\tau(V^G)$ is a linear subshift of V^G.*

3. Invertible linear cellular automata

When A is a finite alphabet, it is well-known that the inverse of a bijective cellular automaton between subshifts of A^G is also a cellular automaton. A topological proof of this result immediately follows from compacity of subshifts and Hedlund's characterization [GoH] of cellular automata as being exactly the continuous maps commuting with the shift action (see Theorem 1.5.14 in [LiM] for an alternative proof). The following is a linear analogue.

Theorem 3.1: *Let G be a countable group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $X, Y \subset V^G$ be linear subshifts and suppose that $\tau\colon X \to Y$ is a bijective cellular automaton. Then the inverse map $\tau^{-1}\colon Y \to X$ is also a linear cellular automaton.*

Proof: Since τ is linear and G-equivariant, it is clear that τ^{-1} is also linear and G-equivariant. Therefore, it suffices to show that the following local property is satisfied by τ^{-1}: there exists a finite subset $\Omega \subset G$ such that, for $y \in Y$, the element $\tau^{-1}(y)(1_G)$ depends only on the restriction of y to Ω. Let us assume that this property is not satisfied.

Let M be a memory set for τ such that $1_G \in M$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} A_n$, $M \subset A_0$ and $A_n \subset A_{n+1}$ for all

$n \in \mathbb{N}$. Let B_n denote the M-**interior** of A_n, that is, $B_n = \{g \in G : gM \subset A_n\}$. Note that $G = \bigcup_{n \in \mathbb{N}} B_n$ and $B_n \subset B_{n+1}$ for all n.

By our non-locality assumption and the linearity of τ^{-1}, there exists, for each $n \in \mathbb{N}$, an element $y_n \in Y$ such that $y_n|_{B_n} = 0$ but $\tau^{-1}(y_n)(1_G) \neq 0$. Setting $x_n = \tau^{-1}(y_n)$, this is equivalent to $x_n(1_G) \neq 0$ and $\tau(x_n)|_{B_n} = 0$.

Consider, for each $n \in \mathbb{N}$, the vector subspace $L_n \subset X|_{A_n}$ defined by $L_n = \mathrm{Ker}(\tau_n)$, where $\tau_n\colon X|_{A_n} \to Y|_{B_n}$ denotes the linear map induced by τ. For $n \leq m$, the restriction map $X|_{A_m} \to X|_{A_n}$ induces a linear map $\pi_{n,m}\colon L_m \to L_n$. Consider now, for all $n \leq m$, the vector subspace $K_{n,m} \subset L_n$ defined by $K_{n,m} = \pi_{n,m}(L_m)$. We have $K_{n,m'} \subset K_{n,m}$ for all $n \leq m \leq m'$ since $\pi_{n,m'} = \pi_{n,m} \circ \pi_{m,m'}$. Hence, if we fix n, the sequence $K_{n,m}$, where $m = n, n+1, \ldots$, is a decreasing sequence of vector subspaces of L_n. Since $L_n \subset X|_{A_n} \subset V^{A_n}$ is finite-dimensional, this sequence stabilizes, i.e., there exists a vector subspace $J_n \subset L_n$ and an integer $k_n \geq n$ such that $K_{n,m} = J_n$ for all $m \geq k_n$.

For all $n \leq n' \leq m$, we have $\pi_{n,n'}(K_{n',m}) \subset K_{n,m}$ since $\pi_{n,n'} \circ \pi_{n',m} = \pi_{n,m}$. Therefore, $\pi_{n,n'}$ induces by restriction a linear map $\rho_{n,n'}\colon J_{n'} \to J_n$ for all $n \leq n'$. We claim that $\rho_{n,n'}$ is surjective. To see this, let $u \in J_n$. Let us choose m large enough so that $J_n = K_{n,m}$ and $J_{n'} = K_{n',m}$. Then we can find $v \in L_m$ such that $u = \pi_{n,m}(v)$. We have $u = \rho_{n,n'}(w)$, where $w = \pi_{n',m}(v) \in K_{n',m} = J_{n'}$. This proves the claim.

Now, using the surjectivity of $\rho_{n,n+1}$ for all n, we construct by induction a sequence of elements $z_n \in J_n$, $n \in \mathbb{N}$, as follows. We start by taking as z_0 the restriction of x_{k_0} to A_0. Observe that $x_{k_0} \in L_{k_0}$ and hence $z_0 = \pi_{0,k_0}(x_{k_0}) \in J_0$. Then, assuming that z_n has been constructed, we take as z_{n+1} an arbitrary element in $\rho_{n,n+1}^{-1}(z_n)$. Since z_{n+1} coincides with z_n on A_n, there exists a unique element $z \in V^G$ such that $z|_{A_n} = z_n$ for all n. We have $z \in X$, since $z_n \in X|_{A_n}$ for all n and X is closed in V^G. As $z(1_G) = z_0(1_G) = x_{k_0}(1_G) \neq 0$, we have $z \neq 0$. On the other hand, $\tau(z) = 0$ since $\tau(z)|_{B_n} = \tau_n(z_n) = 0$ for all n by construction. This contradicts the injectivity of τ. ∎

4. Matrix representation of finite-dimensional linear cellular automata

Let G be a group and $\mathbb{K}$ a field. The aim of this section is to give a matrix description of linear cellular automata $\tau\colon V^G \to V^G$, where V is a vector space of finite dimension d over a field $\mathbb{K}$. Note that, after choosing a basis for V, we can assume $V = \mathbb{K}^d$.

Let $\mathbb{K}[G]$ denote the vector subspace of $\mathbb{K}^G$ consisting of all finitely-supported maps (a map $\alpha\colon G \to \mathbb{K}$ is **finitely supported** if $|\{g \in G : \alpha(g) \neq 0\}| < \infty$). A basis for $\mathbb{K}[G]$ is given by $(\delta_g)_{g\in G}$, where $\delta_g\colon G \to \mathbb{K}$ is defined by $\delta_g(g) = 1$ and $\delta_g(g') = 0$ if $g' \neq g$.

For $x \in \mathbb{K}^G$ and $\alpha \in \mathbb{K}[G]$, the **convolution product** $x\alpha \in \mathbb{K}^G$ is defined by

$$(4.1) \qquad (x\alpha)(g) = \sum_{h\in G} x(h)\alpha(h^{-1}g), \quad \text{for all } g \in G.$$

The convolution product is $\mathbb{K}$-bilinear. We have $\alpha\beta \in \mathbb{K}[G]$ and $(x\alpha)\beta = x(\alpha\beta)$ for all $x \in \mathbb{K}^G$ and $\alpha, \beta \in \mathbb{K}[G]$. Note also that

$$(4.2) \qquad (x\delta_h)(g) = x(gh^{-1}) \quad \text{for all } g \in G,$$

for all $x \in \mathbb{K}^G$ and $h \in G$. In particular, we have $x\delta_{1_G} = x$ and $\delta_g\delta_{g'} = \delta_{gg'}$ for all $x \in \mathbb{K}^G$ and $g, g' \in G$.

The convolution product induces on $\mathbb{K}[G]$ a structure of $\mathbb{K}$-algebra with unit element $1_{\mathbb{K}[G]} = \delta_{1_G}$. This $\mathbb{K}$-algebra is called the **group algebra** of G with coefficients in $\mathbb{K}$. Given $\alpha \in \mathbb{K}[G]$, we define $\overline{\alpha} \in \mathbb{K}[G]$ by $\overline{\alpha}(g) = \alpha(g^{-1})$. We have $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$ for all $\alpha, \beta \in \mathbb{K}[G]$. It follows that the map $\alpha \mapsto \overline{\alpha}$ is an anti-involution of $\mathbb{K}[G]$.

The convolution product naturally extends to matrices. More precisely, let $d \geq 1$ be an integer and let $\mathrm{Mat}_d(\mathbb{K}[G])$ denote the $\mathbb{K}$-algebra of $d \times d$ matrices with coefficients in $\mathbb{K}[G]$. For $x = (x_1, x_2, \ldots, x_d) \in (\mathbb{K}^G)^d$ and $\alpha = (\alpha_{ij})_{i,j=1}^d \in \mathrm{Mat}_d(\mathbb{K}[G])$, we define $x\alpha = (y_1, y_2, \ldots, y_d) \in (\mathbb{K}^G)^d$ by setting

$$y_j = \sum_{i=1}^d x_i\alpha_{ij}$$

for all $j = 1, 2, \ldots, d$, where $x_i\alpha_{ij}$ is the convolution product of $x_i \in \mathbb{K}^G$ and $\alpha_{ij} \in \mathbb{K}[G]$ defined using (4.1).

In the $\mathbb{K}$-algebra $\mathrm{Mat}_d(\mathbb{K}[G])$, we consider the anti-involution $\alpha \mapsto \overline{\alpha}$, where $\overline{\alpha}_{ij} = \overline{\alpha_{ji}}$ for all $i, j = 1, 2, \ldots, d$.

For each $\alpha \in \mathrm{Mat}_d(\mathbb{K}[G])$, let us define the map $\tau_\alpha\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$ by

$$\tau_\alpha(x) = x\overline{\alpha}$$

for all $x = (x_1, x_2, \ldots, x_d) \in (\mathbb{K}^G)^d = (\mathbb{K}^d)^G$. We denote by $\mathrm{supp}(\alpha)$ the **support** of α, that is, the union of the supports of the entries of α.

Proposition 4.1: *For all $\alpha \in \mathrm{Mat}_d(\mathbb{K}[G])$, the map $\tau_\alpha\colon (K^d)^G \to (\mathbb{K}^d)^G$ is a linear cellular automaton (i.e., $\tau_\alpha \in \mathrm{LCA}(G, K^d)$) with memory set $\mathrm{supp}(\alpha)$. Moreover, the map $\Psi\colon \mathrm{Mat}_d(\mathbb{K}[G]) \to \mathrm{LCA}(G, \mathbb{K}^d)$ defined by $\Psi(\alpha) = \tau_\alpha$ is an isomorphism of $\mathbb{K}$-algebras.*

Proof: Let $\alpha \in \mathrm{Mat}_d(\mathbb{K}[G])$. We can write α in the form

$$\alpha = \sum_{m \in M} a_m \delta_m,$$

where $M = \mathrm{supp}(\alpha) \subset G$ and $a_m \in \mathrm{Mat}_d(\mathbb{K})$ for all $m \in M$. For $x \in (\mathbb{K}^d)^G$, we have

$$\tau_\alpha(x) = x\overline{\alpha} = \sum_{m \in M} x\delta_{m^{-1}}\overline{a_m},$$

where $\overline{a_m} \in \mathrm{Mat}_d(\mathbb{K})$ is the transpose of a_m. Using (4.2), we get

$$\tau_\alpha(x)(g) = \sum_{m \in M} u_m(x(gm)) \quad \text{for all } g \in G,$$

where $u_m\colon \mathbb{K}^d \to \mathbb{K}^d$ is the $\mathbb{K}$-linear map $(\lambda_1, \lambda_2, \dots, \lambda_d) \mapsto (\lambda_1, \lambda_2, \dots, \lambda_d)\overline{a_m}$. From the characterization of linear cellular automata given in Lemma 2.3, it follows that τ_α is a linear cellular automaton with memory set M and that Ψ is surjective.

If we take $x = (0, \dots, 0, \delta_{1_G}, 0, \dots, 0)$, where δ_{1_G} is at the i-th component of x, then $\tau_\alpha(x)$ is equal to the i-th row of $\overline{\alpha}$. This shows that Ψ is injective.

Finally, for $\alpha, \beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ and $x \in (\mathbb{K}^d)^G$, we have

$$\tau_{\alpha\beta}(x) = x\overline{\alpha\beta} = x(\overline{\beta}\overline{\alpha}) = (x\overline{\beta})\overline{\alpha} = \tau_\alpha(x\overline{\beta}) = \tau_\alpha(\tau_\beta(x)),$$

so that $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta$. Since τ_α is the identity map on $(\mathbb{K}^d)^G$ for $\alpha = 1_{\mathrm{Mat}_d(\mathbb{K}[G])}$, we conclude that Ψ is an isomorphism of $\mathbb{K}$-algebras. ∎

Remarks: 1) In the one-dimensional case the above proposition reduces to Proposition 6.1 of [CeC].

2) When $\mathbb{K}$ is a finite field, linear cellular automata $\tau_\alpha\colon (\mathbb{K}^d)^{\mathbb{Z}} \to (\mathbb{K}^d)^{\mathbb{Z}}$, $\alpha \in \mathrm{Mat}_d(\mathbb{K}[\mathbb{Z}])$, are called **convolutional encoders** in Section 1.6 of [LiM]. Note that $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[t, t^{-1}]$ is the $\mathbb{K}$-algebra of **Laurent polynomials** with coefficients in $\mathbb{K}$.

Proposition 4.2: *Let $\alpha \in \mathrm{Mat}_d(\mathbb{K}[G])$ and consider the associated linear cellular automaton $\tau_\alpha\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$. Then the following hold:*

(i) τ_α *is injective if and only if there exists* $\beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ *such that* $\beta\alpha = 1$*;*
(ii) *if there exists* $\beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ *such that* $\alpha\beta = 1$ *then* τ_α *is surjective;*
(iii) τ_α *is bijective if and only if* $\alpha \in \mathrm{GL}_d(\mathbb{K}[G])$*.*

Proof: (i) If $\beta\alpha = 1$, then $\tau_\beta \circ \tau_\alpha = \tau_{\beta\alpha}$ is the identity map $I\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$ and therefore τ_α is injective.

Conversely, suppose that τ_α is injective. First we treat the case when G is countable. Then, by Theorem 3.1, there is a linear cellular automaton $\sigma\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$ such that $\sigma \circ \tau_\alpha = I$. We have $\sigma = \tau_\beta$ for some $\beta \in \mathrm{Mat}_d(\mathbb{K}[G])$. Since $\tau_{\beta\alpha} = \tau_\beta \circ \tau_\alpha = I$, we get $\beta\alpha = 1$.

We now treat the general case. Consider the subgroup H of G generated by $\mathrm{supp}(\alpha)$. By Proposition 2.4, the restriction $\tau'\colon (\mathbb{K}^d)^H \to (\mathbb{K}^d)^H$ of τ to H is injective. On the other hand, τ' is also represented by the matrix α in $\mathrm{Mat}_d(\mathbb{K}[H])$. Since H is finitely generated and hence countable, we know from the first case that there exists $\beta \in \mathrm{Mat}_d(\mathbb{K}[H])$ such that $\beta\alpha = 1$. This completes the proof of (i) as $\mathrm{Mat}_d(\mathbb{K}[H]) \subset \mathrm{Mat}_d(\mathbb{K}[G])$.

(ii) If $\alpha\beta = 1$, then $\tau_\alpha \circ \tau_\beta = \tau_{\alpha\beta} = I$ and therefore τ_α is surjective.

(iii) If $\alpha \in \mathrm{GL}_d(\mathbb{K}[G])$, then $\alpha^{-1}\alpha = \alpha\alpha^{-1} = 1$ and τ_α is bijective by (i) and (ii).

Conversely, suppose that τ_α is bijective. Then we can find $\beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ such that $\beta\alpha = 1$ by (i). This implies $\tau_\beta \circ \tau_\alpha = \tau_{\beta\alpha} = I$. Therefore $\tau_\beta = (\tau_\alpha)^{-1}$. It follows that $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta = I$. This implies $\alpha\beta = 1$. Thus $\alpha \in \mathrm{GL}_d(\mathbb{K}[G])$. ∎

Remark: The converse of assertion (ii) in the preceding proposition does not hold. For instance, consider the one-dimensional linear cellular automaton $\tau\colon \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$ defined by $\tau(x)(n) = x(n+1) - x(n)$ for all $x \in \mathbb{K}^{\mathbb{Z}}$, $n \in \mathbb{Z}$. Clearly τ is surjective. However $\tau = \tau_\alpha$ for $\alpha = t - 1 \in \mathbb{K}[\mathbb{Z}] = \mathbb{K}[t, t^{-1}]$, and α is not invertible in $\mathbb{K}[\mathbb{Z}]$.

Corollary 4.3: *The following conditions are equivalent:*
(a) *every injective linear cellular automaton* $\tau\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$ *is surjective;*
(b) *the ring* $\mathrm{Mat}_d(\mathbb{K}[G])$ *is directly finite.*

Proof: (a) $\Rightarrow$ (b). Suppose (a). Let $\alpha, \beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ such that $\beta\alpha = 1$. Then the linear cellular automaton τ_α is injective by Proposition 4.2 (i) and hence bijective by (a). Therefore $\alpha \in \mathrm{GL}_d(\mathbb{K}[G])$ by Proposition 4.2 (iii). But then we can write $\alpha\beta = \alpha(\beta\alpha)\alpha^{-1} = 1$. This shows (b).

(b) ⇒ (a). Suppose (b). Let $\tau\colon (\mathbb{K}^d)^G \to (\mathbb{K}^d)^G$ be an injective linear cellular automaton. Then $\tau = \tau_\alpha$ for some $\alpha \in \mathrm{Mat}_d(\mathbb{K}[G])$ by Proposition 4.1. Since τ is injective, it follows from Proposition 4.2 (i) that there exists $\beta \in \mathrm{Mat}_d(\mathbb{K}[G])$ such that $\beta\alpha = 1$. We have $\alpha\beta = 1$ since $\mathrm{Mat}_d(\mathbb{K}[G])$ is directly finite. Hence $\tau_\alpha \circ \tau_\beta = \tau_{\alpha\beta} = I$. This shows that τ_α is surjective. ∎

Theorem 1.3 follows immediately from Corollary 4.3.

5. L-surjunctivity of sofic groups

In this section we prove that sofic groups are L-surjunctive (Theorem 1.2). In virtue of Definition 2.2 and Corollary 2.5, we can reduce to the case of **finitely generated** groups.

Let G be a finitely generated sofic group with finite symmetric generating subset $S \subset G$. Let V be a vector space of finite dimension $d \geq 1$ over a field $\mathbb{K}$, and let $\tau\colon V^G \to V^G$ be an injective linear cellular automaton. We want to show that τ is surjective.

We shall use the following notation. For $r \in \mathbb{N}$, we denote by $B(r) \subset G$ the ball of radius r centered at 1_G in the Cayley graph associated with (G, S). We set $Y = \tau(X)$. Recall that Y is a linear subshift of V^G by Lemma 2.8.

We can assume that τ has memory set $M = B(r_0)$ for some $r_0 \in \mathbb{N}$. Let $\mu\colon V^{B(r_0)} \to V$ denote the corresponding local defining map.

We choose r_0 large enough so that $\tau^{-1}\colon Y \to V^G$ also admits $B(r_0)$ as a memory set. Let $\nu\colon V^{B(r_0)} \to V$ be a local defining map for τ^{-1}.

We proceed by contradiction. Suppose that τ is not surjective. Then, since Y is closed in V^G by Lemma 2.8, there exists a finite subset $\Omega \subset G$ such that $Y|_\Omega \subsetneq V^\Omega$. It is not restrictive, up to enlarging r_0, to suppose that $\Omega \subset B(r_0)$. Thus, $Y|_{B(r_0)} \subsetneq V^{B(r_0)}$.

Let $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{d|B(2r_0)| + 1}. \tag{5.1}$$

Note that from (5.1) we have $1 - \varepsilon > 1 - \frac{1}{d|B(2r_0)|+1}$ which yields

$$(1-\varepsilon)^{-1} < 1 + \frac{1}{d|B(2r_0)|}. \tag{5.2}$$

Since G is sofic, we can find a finite S-labeled graph (Q, E, λ) such that

$$|Q(3r_0)| \geq (1-\varepsilon)|Q|, \tag{5.3}$$

where we recall that, by definition, $Q(r) = \{q \in Q : q \sim_r 1_G\}$ (see Definition 2.2).

Note the inclusions

$$Q(r_0) \supset Q(2r_0) \supset \cdots \supset Q(ir_0) \supset Q((i+1)r_0) \supset \cdots.$$

For each integer $i \geq 1$, we define the map μ_i: $V^{Q(ir_0)} \to V^{Q((i+1)r_0)}$ by setting, for all $u \in V^{Q(ir_0)}$ and $q \in Q((i+1)r_0)$,

$$\mu_i(u)(q) = \mu(u|_{B(q,r_0)} \circ \phi_q)(1_G),$$

where ϕ_q is the unique isomorphism of S-labeled graphs from $B(r_0) \subset G$ to $B(q, r_0) \subset Q$ sending 1_G to q.

Similarly, we define the map ν_i: $V^{Q(ir_0)} \to V^{Q((i+1)r_0)}$ by setting, for all $u \in V^{Q(ir_0)}$ and $q \in Q((i+1)r_0)$,

$$\nu_i(u)(q) = \nu(u|_{B(q,r_0)} \circ \phi_q)(1_G).$$

From the fact that $\tau^{-1} \circ \tau$ is the identity map on V^G, we deduce that the composite $\nu_{i+1} \circ \mu_i$: $V^{Q(ir_0)} \to V^{Q((i+2)r_0)}$ is the identity on $V^{Q((i+2)r_0)}$. More precisely, denoting by ρ_i: $V^{Q(ir_0)} \to V^{Q((i+2)r_0)}$ the restriction map, we have that $\nu_{i+1} \circ \mu_i = \rho_i$ for all $i \geq 1$. In particular, we have $\nu_2 \circ \mu_1 = \rho_1$. Thus, setting $Z = \mu_1(V^{Q(r_0)}) \subset V^{Q(2r_0)}$, we deduce that $\nu_2(Z) = \rho_1(V^{Q(r_0)}) = V^{Q(3r_0)}$. It follows that

$$\dim(Z) \geq d|Q(3r_0)|. \tag{5.4}$$

In order to estimate the dimension of Z from above, we need the following lemma.

Lemma 5.1: *There exists a subset $Q' \subset Q(3r_0)$ satisfying*

$$|Q'| \geq \frac{|Q(3r_0)|}{|B(2r_0)|} \tag{5.5}$$

such that the balls $B(q', r_0)$ with $q' \in Q'$ are all disjoint.

Proof: Let Q' be a maximal subset of $Q(3r_0)$ such that the balls $B(q', r_0)$ with $q' \in Q'$ are all disjoint. If $q \in Q(3r_0) \setminus Q'$ is at distance greater than $2r_0$ from Q', then $B(q, r_0) \cap B(q', r_0) = \varnothing$ for all $q' \in Q'$, contradicting the maximality of Q'. Therefore $Q(3r_0)$ is contained in the union of the balls $B(q', 2r_0)$, $q' \in Q'$. This implies

$$|Q(3r_0)| \leq |Q'| \cdot |B(2r_0)|,$$

which gives (5.5). ∎

Let $Q' \subset Q(3r_0)$ be as in the preceding lemma and set $\overline{Q'} = \coprod_{q' \in Q'} B(q', r_0)$. Note that $\overline{Q'} \subset Q(2r_0)$ so that

$$|Q(2r_0)| = |Q'| \cdot |B(r_0)| + |Q(2r_0) \setminus \overline{Q'}|. \tag{5.6}$$

Now observe that, for all $q \in Q(2r_0)$, we have a natural isomorphism of vector spaces $Z|_{B(q,r_0)} \to Y|_{B(r_0)}$ given by $u \mapsto u \circ \phi_q$, where ϕ_q denotes, as above, the unique isomorphism of S-labeled graphs from $B(r_0)$ to $B(q, r_0)$ sending 1_G to q. Since $Y|_{B(r_0)} \subsetneq V^{B(r_0)}$, this implies that

$$\dim(Z|_{B(q,r_0)}) = \dim(Y|_{B(r_0)}) \leq d \cdot |B(r_0)| - 1, \tag{5.7}$$

for all $q \in Q'$.

Thus we have

$$\begin{aligned}
\dim(Z) &\leq \dim(Z|_{\overline{Q'}}) + \dim(Z|_{Q(2r_0)\setminus\overline{Q'}}) \\
&\leq |Q'| \cdot (d \cdot |B(r_0)| - 1) + d \cdot |Q(2r_0) \setminus \overline{Q'}| \\
&= d\Big(|Q(2r_0)| - \frac{|Q'|}{d}\Big),
\end{aligned}$$

where the last equality follows from (5.6). Comparing this with (5.4) we obtain

$$|Q(3r_0)| \leq |Q2r_0)| - \frac{|Q'|}{d}.$$

Thus,

$$\begin{aligned}
|Q| \geq |Q(2r_0)| &\geq |Q(3r_0)| + \frac{|Q'|}{d} \\
&\geq |Q(3r_0)| + \frac{|Q(3r_0)|}{d|B(2r_0)|} \quad \text{by (5.5)}, \\
&= |Q(3r_0)|\Big(1 + \frac{1}{d|B(2r_0)|}\Big) \\
&> |Q(3r_0)|(1 - \varepsilon)^{-1}
\end{aligned}$$

where the last inequality follows from (5.2). This yields

$$|Q(3r_0)| < (1 - \varepsilon)|Q|$$

which contradicsts (5.3). This ends the proof of Theorem 1.2.

Acknowledgements: We express our deep gratitude to Fabrice Krieger for pointing an inaccuracy in the proof of Theorem 1.2. We are also very greatful to Benjy Weiss for addressing our attention to the most interesting recent results of Elek and Szabó on sofic groups, and to Gabor Elek for sending us all these (p)reprints and for suggesting that an alternative proof of his joint result with Szabó on stable finiteness of sofic groups could be deduced from our L-surjunctivity theorem for such groups.

References

[CeC] T. Ceccherini-Silberstein and M. Coornaert, *The Garden of Eden theorem for linear cellular automata,* Ergodic Theory and Dynamical Systems **26** (2006), 53–68.

[ES1] G. Elek and A. Szabó, *Sofic groups and direct finiteness,* Journal of Algebra **280** (2004), 426–434.

[ES2] G. Elek and A. Szabó, *On sofic groups,* Journal of Group Theory **9** (2006), 161–171.

[Got] W. Gottschalk, *Some general dynamical systems,* Recent Advances in Topological Dynamics, Lecture Notes in Mathematics **318**, Springer, Berlin, 1973, pp. 120–125.

[GoH] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics,* American Mathematical Society Colloquium Publications, Vol. 36. American Mathematical Society, Providence, R.I., 1955.

[Gro] M. Gromov, *Endomorphisms of symbolic algebraic varieties,* Journal of the European Mathematical Society **1** (1999), 109–197.

[K] I. Kaplansky, *Fields and Rings,* Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1969.

[LiM] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding,* Cambridge University Press, Cambridge, 1995.

[Wei] B. Weiss, *Sofic groups and dynamical systems,* (Ergodic theory and harmonic analysis, Mumbai, 1999) The Indian Journal of Statistics. Series A **62** (2000), 350–359.